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Entropy functional of a non-uniform Ising model on a two-row lattice

M Q Zhang
Courant Institute of Mathematical Sciences, New York University, New York, NY 10012, USA

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Abstract. An exact entropy functional is derived for an Ising model on a $2 \times N$ lattice with arbitrary vertical bonds and arbitrary external fields. Explicit results for special cases are discussed.

1. Introduction

Increasing attention has been paid to the study of inhomogeneous models, ranging from 1D classical Ising models [1–4] and continuous hard rod fluids [5], to special 2D vertex models [6] and interface models [7, 8]. More encouraging results were obtained in the so-called inverse problem initiated by Percus [1] and generalised to inhomogeneous couplings by Tejero [9]. It consists in finding the external fields needed to produce a given magnetisation profile (i.e. the inverse profile). It has been further extended to the Bethe lattice [10] and lattices with articulation points [11] by Šamaj. Despite many successes, the inverse technique has long been restricted to special topologies which do not have feedback loops, since only in this case are the inverse solutions local. Otherwise, there exist collective modes [12]. Recently, Percus and Zhang solved the inhomogeneous Ising model on a ring [13] and then on a multi-connected network [14] by enlarging the phase space to include collective mode variables. Because this approach gets increasingly involved as more and more loops are introduced, it cannot be used in higher dimensions. Even the problems of inhomogeneous Ising model with next-nearest-neighbour interaction or with higher spins on a ring are still open (although some partial results for next-nearest-neighbour hard core exclusion have been obtained [15]). The purpose of the current investigation is to demonstrate the possibility of generalising 1D density functional to higher dimensions. This will be done by presenting exactly solvable 2D examples where one can analyse deeper aspects of the inverse problem of statistical mechanics, which reads in its most general form: find coupling constants from correlation functions.

In this paper, we study an inhomogeneous Ising model on a $2 \times N$ lattice with arbitrary vertical couplings and external fields by applying the idea of the entropy functional [16, 17], which was introduced originally for a continuous system. It can be regarded as a generalisation of the original inverse method to the problem of determining both internal and external interactions needed to evoke given magnetisations and nearest-neighbour pair correlation functions. In section 2, we shall set up the model and introduce the concept of the entropy functional. In section 3, we review the basic formulation but in a general context for the class of models of this type. In section 4, we apply the general formulation to our problem, and derive the equations which govern the inverse profile. In the last section, we study the properties of the solutions and discuss some special cases (explicit solutions on some hypersurfaces). The major property which is common to all cases, and which distinguishes the entropy formulation of this system, is that the locality of the interaction is mirrored by locality of magnetisation profile and thermodynamic potentials.

2. Model set-up and entropy functional

Consider the Ising model with spin variables (one may describe the system as a lattice gas by the standard transformation $\sigma \to 2\nu - 1$) $\sigma_x, \bar{\sigma}_x = \pm 1$ defined on a $2 \times N$ lattice as shown in figure 1.

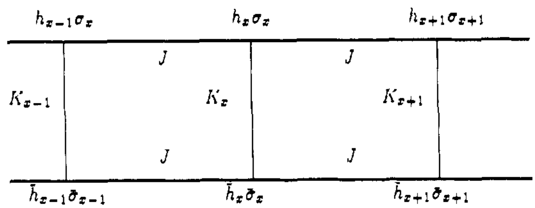

Figure 1. The two-row Ising lattice.

Since we are not concerned with boundary effects here, we may take N to be very large. The partition function is the standard one

$$Z[\boldsymbol{h}, \bar{\boldsymbol{h}}, \boldsymbol{K}] = \sum_{\{\sigma_x, \bar{\sigma}_x\}} \exp\left(J \sum (\sigma_x \sigma_{x+1} + \bar{\sigma}_x \bar{\sigma}_{x+1}) + \sum K_x \sigma_x \bar{\sigma}_x + \sum (h_x \sigma_x + \bar{h}_x \bar{\sigma}_x) \right) \tag{1}$$

with $h_x, \bar{h}_x$ being the external fields and $K_x(J)$ the vertical (horizontal) couplings, and we have set $\beta = 1$. Here $h_x, \bar{h}_x$ and K_x are treated as arbitrary variables. The free energy $F[\boldsymbol{h}, \bar{\boldsymbol{h}}, \boldsymbol{K}] = -\ln Z$ is the generating functional of the magnetisations $m_x = \langle \sigma_x \rangle, \bar{m}_x = \langle \bar{\sigma}_x \rangle$ and the vertical pair correlations $c_x = \langle \sigma_x \bar{\sigma}_x \rangle$

$$m_x = -\frac{\partial F}{\partial h_x} \qquad \bar{m}_x = -\frac{\partial F}{\partial \bar{h}_x} \qquad c_x = -\frac{\partial F}{\partial K_x}. \tag{2}$$

The Legendre transform of F

$$-S[\boldsymbol{m}, \bar{\boldsymbol{m}}, \boldsymbol{c}] = F + \sum (h_x m_x + \bar{h}_x \bar{m}_x + K_x c_x) \tag{3}$$

is the so-called entropy functional [16] (up to a constant $\sim 2NJ$). Regarding S as a functional of the densities shown in its arguments, we then find from (2) and (3) the associated conjugate relations

$$h_x = -\frac{\partial S}{\partial m_x} \qquad \bar{h}_x = -\frac{\partial S}{\partial \bar{m}_x} \qquad K_x = -\frac{\partial S}{\partial c_x}. \tag{4}$$

The task is to construct S (a generalised inverse problem), Equations (4) constitute a complete solution in inverse form of the profile problem at the one- and (nearest-neighbour) two-body levels, which contains the full thermodynamics.

3. General formulation

The method of constructing S and the inverse profile equation is a discrete analogy of [16], which is just a straightforward generalisation of [2] to higher dimensions. Let $T_x(v_x, v'_x)$, the transfer matrix with v_x the microscopic configuration in the hyperplane orthogonal to the x direction, have the following form

$$T_x(v, v') = W_x^{1/2}(v)\,\mathbf{e}(v, v')\,W_{x+1}^{1/2}(v')$$

with a matrix $\mathbf{e}(v, v')$ describing some interaction. We define two vectors as

$$Z_x^- = W_x^{1/2} T_x T_{x+1} \cdots |R\rangle Z^{-1} \qquad Z_x^+ = Z^{-1} \langle L| \cdots T_{x-2} T_{x-1} W_x^{1/2} \tag{5}$$

where $\langle L|$ and $|R\rangle$ are the left and right boundary state vectors, respectively. Then we have the following pair of recurrence relations

$$Z_x^- = W_x\,\mathbf{e}\,Z_{x+1}^- \qquad Z_x^+ = Z_{x-1}^+\,\mathbf{e}\,W_x\,. \tag{6}$$

By definition, the probability for the configuration $v_x = v$ is given by

$$n_x(v) = Z_x^+(v) W_x^{-1} Z_x^-(v)\,. \tag{7}$$

By eliminating W in (6) and (7), one arrives a pair of nonlinear equations for the ratios of $Z_x^\pm(v)/Z_x^\pm(v')$

$$\begin{aligned} &\sum_{v',v''} \mathbf{e}^{\mathrm{T}}(v, v') \frac{Z_x^-(v')}{Z_x^-(v'')} n_x(v'') (\mathbf{e}^{\mathrm{T}})^{-1}(v'', v) = n_{x-1}(v) \\ &\sum_{v',v''} \mathbf{e}(v, v') \frac{Z_x^+(v')}{Z_x^+(v'')} n_x(v'') \mathbf{e}^{-1}(v'', v) = n_{x+1}(v) \end{aligned} \tag{8}$$

where $\mathbf{e}^{\mathrm{T}}$ is the transpose of $\mathbf{e}$. The above expressions uniquely determine the ratios, provided the obvious normalisation

$$\sum_v n_x(v) = 1 \tag{9}$$

is added. Once the ratios are obtained, the solution for the inverse profile is then given, following (7), by

$$\frac{W_x(v)}{W_x(v')} = \frac{Z_x^+(v) Z_x^-(v) n_x(v')}{Z_x^+(v') Z_x^-(v') n_x(v)}\,. \tag{10}$$

From the general formulation above, we learn that, with a suitable enlargement of the density space (including as well the appropriate correlation functions), it is also possible to extend the techniques of inverse solutions of 1D inhomogeneous systems to higher dimensions.

4. Explicit equations

The general formulation, applied to the current situation, amounts to making the following identities:

$$v_x = \{\sigma_x, \bar{\sigma}_x\}$$
$$W_x(v_x) = \exp(h_x\sigma_x + \bar{h}_x\bar{\sigma}_x + K_x\sigma_x\bar{\sigma}_x)$$
$$\mathbf{e}(v, v') = \exp(J\sigma\sigma' + J\bar{\sigma}\bar{\sigma}') .$$

Then, the $m_x, \bar{m}_x, c_x$ and $n_x(v)$ are related by

$$m_x = \sum_{\sigma_x\bar{\sigma}_x} \sigma_x n_x(\sigma_x, \bar{\sigma}_x)$$

$$\bar{m}_x = \sum_{\sigma_x\bar{\sigma}_x} \bar{\sigma}_x n_x(\sigma_x, \bar{\sigma}_x) \tag{11}$$

$$c_x = \sum_{\sigma_x\bar{\sigma}_x} \sigma_x\bar{\sigma}_x n_x(\sigma_x, \bar{\sigma}_x) \tag{12}$$

or equivalently we may express $n_i, i = 1, \ldots, 4$(defined below) in terms of $m, \bar{m}$ and c as

$$\begin{aligned} n_1 &\equiv n_x(++) = \tfrac{1}{4}(1 + c_x + m_x + \bar{m}_x) \\ n_2 &\equiv n_x(+-) = \tfrac{1}{4}(1 - c_x + m_x - \bar{m}_x) \\ n_3 &\equiv n_x(-+) = \tfrac{1}{4}(1 - c_x - m_x + \bar{m}_x) \\ n_4 &\equiv n_x(--) = \tfrac{1}{4}(1 + c_x - m_x - \bar{m}_x) \end{aligned} \tag{13}$$

(hereafter, for convenience, we number the four possible states at each site in a natural way as shown above; this should not be confused with the value of x which we shall suppress whenever possible).

Since our matrix $\mathbf{e}$ is symmetric, we can rewrite (8) in its generic form

$$\sum_{i \neq j} e_{ki} \frac{Y_i}{Y_j} n_j e_{jk}^{-1} = b_k \tag{14}$$

where we have combined the diagonal sum with the right hand side. Y_i can be regarded as either of $Z_x(i)^{\pm}$ and it is understood that b should correspondingly be thought of as $b^{\pm}$. With $\mathbf{e}$ given by ($\epsilon \equiv \exp(2J)$)

$$\mathbf{e} = \begin{pmatrix} \epsilon & 1 & 1 & \epsilon^{-1} \\ 1 & \epsilon & \epsilon^{-1} & 1 \\ 1 & \epsilon^{-1} & \epsilon & 1 \\ \epsilon^{-1} & 1 & 1 & \epsilon \end{pmatrix} \qquad \mathbf{e}^{-1} = (\epsilon - \epsilon^{-1})^{-2} \begin{pmatrix} \epsilon & -1 & -1 & \epsilon^{-1} \\ -1 & \epsilon & \epsilon^{-1} & -1 \\ -1 & \epsilon^{-1} & \epsilon & -1 \\ \epsilon^{-1} & -1 & -1 & \epsilon \end{pmatrix} \tag{15}$$

the explicit forms of $b_k^{\mp}$ are

$$b_1^{\mp} = n_1^{\mp} - \frac{n_1\epsilon^2 - n_3 - n_2 + n_4\epsilon^{-2}}{(\epsilon - \epsilon^{-1})^2}$$
$$b_2^{\mp} = n_2^{\mp} - \frac{n_2\epsilon^2 - n_4 - n_1 + n_3\epsilon^{-2}}{(\epsilon - \epsilon^{-1})^2}$$
$$b_3^{\mp} = n_3^{\mp} - \frac{n_3\epsilon^2 - n_4 - n_1 + n_2\epsilon^{-2}}{(\epsilon - \epsilon^{-1})^2}$$
$$b_4^{\mp} = n_4^{\mp} - \frac{n_4\epsilon^2 - n_3 - n_2 + n_1\epsilon^{-2}}{(\epsilon - \epsilon^{-1})^2}$$

where $n_i^{\mp} \equiv n_{x\pm 1}(i)$. It is obvious, from (9) that $\sum_i b_i = 0$.

It turns out to be more convenient to define variables ρ_i and functions $\theta_i^{\mp}(\rho)$ as follows:

$$\rho_i \equiv n_i^{1/2} \qquad \theta_i^{\mp} \equiv \ln \frac{Z_i^{\mp}\rho_1}{Z_1^{\mp}\rho_i} \tag{16}$$

and the expressions in (14) become

$$a_2^{\mp} = \rho_1\rho_2 \sinh(\theta_2^{\mp}) + \rho_3\rho_4 \sinh(\theta_4^{\mp} - \theta_3^{\mp}) \tag{17}$$
$$a_3^{\mp} = \rho_1\rho_3 \sinh(\theta_3^{\mp}) + \rho_2\rho_4 \sinh(\theta_4^{\mp} - \theta_2^{\mp}) \tag{18}$$
$$a^{\pm} = 2\cosh(2J)\rho_1(\rho_2 \sinh(\theta_2^{\mp}) + \rho_3 \sinh(\theta_3^{\mp})) + \rho_1\rho_4 \cosh(\theta_4^{\mp}) - \rho_2\rho_3 \cosh(\theta_2^{\mp} - \theta_3^{\mp}) \tag{19}$$

where the a are given by

$$a_2^{\mp} \equiv \frac{b_3^{\mp} - b_2^{\mp} + b_1^{\mp} - b_4^{\mp}}{4(\epsilon - \epsilon^{-1})} = \frac{\bar{m}_{x\pm 1} \sinh 2J - \bar{m}_x \cosh 2J}{2}$$
$$a_3^{\mp} \equiv \frac{b_2^{\mp} - b_3^{\mp} + b_1^{\mp} - b_4^{\mp}}{4(\epsilon - \epsilon^{-1})} = \frac{m_{x\pm 1} \sinh 2J - m_x \cosh 2J}{2} \tag{20a}$$

$$a^{\pm} \equiv \frac{(b_1^{\mp} + b_4^{\mp})(\epsilon - \epsilon^{-1}) + (b_1^{\mp} - b_4^{\mp})(\epsilon + \epsilon^{-1})}{4(\epsilon - \epsilon^{-1})}$$
$$= \frac{c_{x\pm 1} \sinh 2J + (m_{x\pm 1} + \bar{m}_{x\pm 1}) \cosh 2J}{2} \sinh 2J - \frac{m_x + \bar{m}_x + c_x}{2} \cosh^2 2J\,. \tag{20b}$$

Equations (17)–(19) are the basic equations for our system. Given the magnetisations and vertical correlations $\{m_x, \bar{m}_x, c_x\}$, the ρ_i are determined by (13) and (16) and the a by (20*b*). Thus (17)–(19) can be solved for $\theta_i^{\pm}$, after which the inverse profile is found from (10), by

$$h_x = \frac{\Theta_2 - \Theta_3 - \Theta_4}{4}$$
$$\bar{h}_x = \frac{\Theta_3 - \Theta_2 - \Theta_4}{4} \tag{21}$$

$$K_x = \frac{\Theta_4 - \Theta_2 - \Theta_3}{4} \tag{22}$$

where $\Theta_i \equiv \theta_i^+ + \theta_i^-$, and the entropy functional is given by [15]

$$- S[\boldsymbol{m}, \bar{\boldsymbol{m}}, c] = S^0[m, c] + \sum_x \Delta c_x \int_0^1 \mathrm{d}\lambda\, K_x(\boldsymbol{m}^\lambda, \bar{\boldsymbol{m}}^\lambda, c^\lambda) + \sum_x \Delta m_x \int_0^1 \mathrm{d}\lambda\, h_x(\boldsymbol{m}^\lambda, \bar{\boldsymbol{m}}^\lambda, c^\lambda) + \sum_x \Delta \bar{m}_x \int_0^1 \mathrm{d}\lambda\, \bar{h}_x(\boldsymbol{m}^\lambda, \bar{\boldsymbol{m}}^\lambda, c^\lambda) \qquad (23)$$

where S_0 is the entropy of some uniform reference system, $\Delta f_x \equiv f_x - f_0$, $f_x^\lambda \equiv f_x + \lambda \Delta f_x$ (f_0 is the corresponding uniform value).

5. Solution of the inverse profile equations

The outstanding property of (22)–(23), that is somewhat concealed by the formalism, is that they are local: the profile $\{h_x, \bar{h}_x, k_x\}$ at x depends only upon m_x, $m_{x\pm1}$, $\bar{m}_x$, $\bar{m}_{x\pm1}$, c_x, $c_{x\pm1}$, and one can define an entropy at x in terms of these quantities alone. However, (17)–(19) are strongly coupled algebraic equations of the $\sinh(\theta_i)$. After elimination, one would end up with an irreducible nonlinear equation of high degree which cannot be solved explicitly in closed form, although it can be solved to any required degree of accuracy by methods of successive approximation. For the purpose of illustration, we shall solve the equations in some special cases. The general properties of the equations are:

(i) if all the magnetisations change their signs, so do the external fields;

(ii) if all $m, \bar{m}, c$ are small, so are $h, \bar{h}, K$;

(iii) equations are invariant under the exchange: index $2 \leftrightarrow 3$ and $h, m \leftrightarrow \bar{h}, \bar{m}$ (the symmetry of interchanging rows).

Due to this last property, we only have to solve for $\theta_2^\pm, \theta_4^\pm$ (i.e. h_x, K_x), $\theta_3^\pm$ ($\bar{h}_x$) can be obtained by this interchange of the variables.

5.1. Low densities

From property (ii) mentioned above, we know that this corresponds to small fields (external $h, \bar{h}$ and internal K), and therefore to the small θ (see (22)). To lowest order, we can linearise (17)–(19), and immediately obtain

$$\begin{aligned}\theta_2^\pm &\simeq \frac{(a^\pm + \rho_2\rho_3 - \rho_1\rho_4)(\rho_1\rho_3 + \rho_2\rho_4) + 2\cosh 2J(\rho_2 a_2^\mp - \rho_3 a_3^\mp)\rho_1}{2\cosh 2J \rho_1\rho_2(\rho_2 + \rho_3)(\rho_1 + \rho_4)} \\ &\simeq \sinh 2J \bar{m}_{x\pm1} - \cosh 2J \bar{m}_x + \frac{\sinh^2 2J c_{x\pm1} - (1 + \cosh^2 2J) c_x}{2\cosh 2J} \\ \theta_4^\pm &\simeq \frac{(a^\pm + \rho_2\rho_3 - \rho_1\rho_4)(\rho_4 - \rho_1) + 2\cosh 2J(a_2^\mp - a_3^\mp)\rho_1}{2\cosh 2J \rho_1\rho_4(\rho_2 + \rho_3)} \\ &\simeq \sinh 2J(\bar{m}_{x\pm1} + m_{x\pm1}) - \cosh 2J(m_x + \bar{m}_x)\end{aligned}$$

hence the fields are

$$\begin{aligned} h_x &\simeq \cosh 2J m_x - \sinh 2J \frac{m_{x+1} + m_{x-1}}{2} \\ \bar{h}_x &\simeq \cosh 2J \bar{m}_x - \sinh 2J \frac{\bar{m}_{x+1} + \bar{m}_{x-1}}{2} \\ K_x &\simeq \frac{\cosh 2J + (\cosh 2J)^{-1}}{2} c_x - \frac{\sinh^2 2J}{4\cosh 2J}(c_{x+1} + c_{x-1}). \end{aligned} \qquad (24)$$

We see that for low densities, the 'conjugate' quantities are proportional to one another with the nearest-neighbour corrections. The entropy functional S, from (3), is just

$$-S = -S^0 + \sum_x \Delta c_x \left(\frac{\cosh 2J + (\cosh 2J)^{-1}}{2} c_x - \frac{\sinh^2 2J}{4\cosh 2J}(c_{x+1} + c_{x-1}) \right) + \sum_x \Delta m_x \left(\cosh 2J m_x - \sinh 2J \frac{m_{x+1} + m_{x-1}}{2} \right) + \sum_x \Delta \bar{m}_x \left(\cosh 2J \bar{m}_x - \sinh 2J \frac{\bar{m}_{x+1} + \bar{m}_{x-1}}{2} \right).$$

As in any (Gaussian) linear equation of motion approximation, the 'energy' S is quadratic in densities.

5.2. *The case when* $\theta_4^\pm = \theta_2^\pm + \theta_3^\pm$

First, we suppose this to hold only at x. According to (22), this is the case when $K_x = 0$. To find h_x, we have to solve (17) and (18) for $\theta_3^\pm$. This can be easily done and is given by

$$\theta_3^\pm = \frac{a_3^\pm}{\rho_1\rho_3 + \rho_2\rho_4}$$

from which

$$\begin{aligned} h_x &= -\frac{1}{2}\Theta_3 = -\frac{1}{2}\left[\sinh^{-1}\left(\frac{a_3^+}{\rho_1\rho_3 + \rho_2\rho_4}\right) + \sinh^{-1}\left(\frac{a_3^-}{\rho_1\rho_3 + \rho_2\rho_4}\right)\right] \\ &= \frac{1}{2}\left[\sinh^{-1}\left(\frac{m_x \cosh 2J - m_{x+1}\sinh 2J}{D_x}\right)\right. \\ &\quad \left. + \sinh^{-1}\left(\frac{m_x \cosh 2J - m_{x-1}\sinh 2J}{D_x}\right)\right] \end{aligned} \tag{25}$$

where

$$D_x = \tfrac{1}{2}\left\{[(1+\bar{m}_x)^2 - (c_x + m_x)^2]^{1/2} + [(1-\bar{m}_x)^2 - (c_x - m_x)^2]^{1/2}\right\}. \tag{26}$$

It is very instructive to compare this result with previous ones [14]. When $K_x = 0$, there is a collective mode K 'running' through σ_x which is the same as $\bar{K}$, the 'conjugate' collective mode through $\bar{\sigma}_x$(see [14] for details) and h_x is related to K by

$$h_x = \frac{1}{2}\ln\left[\frac{-2a_3^+ + ((2a_3^+)^2 - m_x^2 + 1 - 4K)^{1/2}}{2a_3^- + ((2a_3^-)^2 - m_x^2 + 1 - 4K)^{1/2}}\right]. \tag{27}$$

Comparing with (25), with the help of (20) and (26), we see that the collective mode K is given by

$$\begin{aligned} 4K &= 1 - m_x - 4(\rho_1\rho_3 + \rho_2\rho_4)^2 \\ &= 1 - m_x - D_x^2 = 1 - \bar{m}_x - \bar{D}_x^2 \\ &= 1 - \bar{m}_x - 4(\rho_1\rho_2 + \rho_3\rho_4)^2 \\ &= \frac{1}{2}\{1 + c_x^2 - m_x^2 - \bar{m}_x^2 \\ &\quad - [((1+\bar{m}_x)^2 - (c_x + m_x)^2)((1-\bar{m}_x)^2 - (c_x - m_x)^2)]^{1/2}\}. \end{aligned} \tag{28}$$

This can be viewed as the hypersurface (equivalent to (19)) on which $K_x = 0$ is realised. It has a rather clear physical meaning: this is the surface on which the 'conjugate' collective modes at x are equal!

If $K_x = 0$ for all x, then we know $K = \bar{K} = 0$. The hypersurface equation (28) becomes local in magnetisation

$$2(\rho_1\rho_3 + \rho_2\rho_4) = D_x = (1 - m_x^2)^{1/2}. \tag{29}$$

Intuitively, this must of course be true, because the whole system 'denatures', separating into two 'strands'. Statistical independence means $n_x(\sigma_x, \bar{\sigma}_x) = \frac{1}{4}(1 + \sigma_x m_x)(1 + \bar{\sigma}_x \bar{m}_x)$; (29) can therefore be verified directly (it is interesting to verify the same through (19)).

5.3. *The case when $\theta_4^{\pm} = 0$ for all x*

This last case is the one with a hypersurface defined by $h_x = \bar{h}_x$ for all x. There are two possibilities.

If $n_1 \neq n_4$, then $h_x \neq 0$. We find, from (17) and (18), that

$$\begin{aligned}\theta_2^{\pm} &= \sinh^{-1} \frac{\rho_1 a_2^{\pm} + \rho_4 a_3^{\pm}}{\rho_2(\rho_1^2 - \rho_4^2)} \\ &= \sinh^{-1}\left(\frac{(\bar{m}_{x\pm 1}\sinh 2J - \bar{m}_x\cosh 2J)(1 + c_x + m_x + \bar{m}_x)^{1/2}}{(m_x + \bar{m}_x)(1 - c_x + m_x - \bar{m}_x)^{1/2}}\right. \\ &\quad \left. + \frac{(m_{x\pm 1}\sinh 2J - m_x\cosh 2J)(1 + c_x - m_x - \bar{m}_x)^{1/2}}{(m_x + \bar{m}_x)(1 - c_x + m_x - \bar{m}_x)^{1/2}}\right).\end{aligned}$$

Of course, the fields are given by (22) and the hypersurface of the solution is given by (19).

If $n_1 = n_4$, then $h_x = \bar{h}_x = 0$ or $\theta_2^{\pm} = \theta_3^{\pm}$. We find, from (19), that

$$\theta_2^{\pm} = \sinh^{-1} \frac{c_{x\pm 1}\sinh^2 2J - c_x(\cosh^2 2J + 1)}{2\cosh 2J(1 - c_x^2)^{1/2}}.$$

The inhomogeneous internal (vertical) coupling field is then given, according to (22), by

$$\begin{aligned}K_x = \frac{1}{2}\left(\sinh^{-1} \frac{c_x(1 + \cosh^2 2J) - c_{x+1}\sinh^2 2J}{2\cosh 2J(1 - c_x^2)^{1/2}}\right. \\ \left. + \sinh^{-1} \frac{c_x(1 + \cosh^2 2J) - c_{x-1}\sinh^2 2J}{2\cosh 2J(1 - c_x^2)^{1/2}}\right).\end{aligned} \tag{30}$$

Interestingly enough, in the uniform system, it reduces to

$$K = \sinh^{-1} \frac{c}{\cosh 2J(1 - c^2)^{1/2}}$$

which translates into the well known result

$$c^2 = \frac{\sinh^2 K \cosh^2 2J}{1 + \sinh^2 K \cosh^2 2J}.$$

For the inhomogeneous system with zero external fields, the entropy functional has only the internal energy part (since $S^0[c=0]=0$ and we have dropped the trivial horizontal part $\sim 2NJ$):

$$S[c]=\frac{1}{2}\sum_x c_x\int_0^1 \{\lambda c_x(1+\cosh^2 2J)-\lambda c_{x+1}\sinh^2 2J$$
$$+[(\lambda c_x(1+\cosh^2 2J)-\lambda c_{x+1}\sinh^2 2J)^2+4(1-\lambda^2 c_x^2)\cosh^2 2J]^{1/2}\}$$
$$\{\lambda c_{x-1}\sinh^2 2J-\lambda c_x(1+\cosh^2 2J)$$
$$+[(\lambda c_x(1+\cosh^2 2J)-\lambda c_{x-1}\sinh^2 2J)^2+4(1-\lambda^2 c_x^2)\cosh^2 2J]^{1/2}\}^{-1}$$
$$=\sum_x c_x K_x+\frac{1}{2}\sum_x\left[\ln\left(\frac{c_x^2-\alpha_x^2}{c_x-\beta_x^2}\right)^{1/2}\frac{(1+\alpha_x^2-c_x^2)^{1/2}+\alpha_x}{(1+\beta_x^2-c_x^2)^{1/2}-\beta_x}\right.$$
$$\left.\times\frac{c_x+\beta_x^2-c_x^2-\beta_x(1+\beta_x^2-c_x^2)^{1/2}}{c_x+\alpha_x^2-c_x^2+\alpha_x(1+\alpha_x^2-c_x^2)^{1/2}}-\ln\left(\left(\frac{(c_x-\alpha_x)(c_x-\beta_x)}{(c_x+\alpha_x)(c_x+\beta_x)}\right)^{1/2}\right)\right]$$

where

$$\alpha_x\equiv\frac{c_x(1+\cosh^2 2J)-c_{x+1}\sinh^2 2J}{2\cosh 2J}$$
$$\beta_x\equiv\frac{c_x(1+\cosh^2 2J)-c_{x-1}\sinh^2 2J}{2\cosh 2J}.$$

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References

[1] Percus J K 1977 *J. Stat. Phys.* **16** 299
[2] Percus J K 1982 *J. Math. Phys.* **23** 1162
[3] Mendès-France M 1986 *J. Stat. Phys.* **45** 89
[4] Derrida B, Mendès-France M and Peyrière J 1986 *J. Stat. Phys.* **45** 439
[5] Percus J K 1976 *J. Stat. Phys.* **15** 505
[7] Robert M and Widom B 1984 *J. Stat. Phys.* **37** 419
[6] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (New York: Academic)
[8] Ord G, Percus J K and Zhang M Q 1988 *Nucl. Phys.* **305** 271
[9] Tejero C F 1987 *J. Stat. Phys.* **48** 531
[10] Šamaj L 1988 *Physica* A **153** 517
[11] Šamaj L 1989*J. Physique* **50** 273
[12] Percus J K 1989 *J. Stat. Phys.* **55** 1263
[13] Percus J K and Zhang M Q 1988 *Phys. Rev.* B **38** 11737
[14] Zhang M Q and Percus J K 1989 *J. Stat. Phys.* **56** 695
[15] Zhang M Q and Percus J K 1990 to be published
[16] Percus J K 1989 *J. Phys: Condens. Matter* 1 2911
[17] Percus J K 1990 *J. Stat. Phys.* at press